A CLASSIFICATION OF CENTRALLY-SYMMETRIC AND CYCLIC 12-VERTEX TRIANGULATIONS OF $S^2 \times S^2$

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Abstract. In this paper our main result states that there exist exactly three combinatorially distinct centrally-symmetric 12-vertex-triangulations of the product of two 2-spheres with a cyclic symmetry. We also compute the automorphism groups of the triangulations. These instances suggest that there is a triangulation of $S^2 \times S^2$ with 11 vertices – the minimum number of vertices required.

1. Introduction and Main Theorem

By a theorem of Kühnel [Kü1] any triangulation of $S^2 \times S^2$ must have at least 10 vertices and any triangulation with only 10 vertices must be 3-neighborly, i. e. any 3-tuple of vertices spans a 2-simplex of the triangulation. By a computer-aided enumeration Kühnel and the first author [K-L] proved that there does not exist a 3-neighborly triangulation of a 4-manifold with 10 vertices, hence any triangulation of $S^2 \times S^2$ requires 11 vertices or more. Up to now no triangulation of $S^2 \times S^2$ with only 11 vertices has been published (cf. however Remark 4.2), but the second author has found in [Sp1] a highly-symmetric 12-vertex-triangulation of $S^2 \times S^2$. In this paper we present a classification of all centrally-symmetric 12-vertex-triangulations of $S^2 \times S^2$ with a cyclic symmetry. We will establish that one of the triangulations of this classification is combinatorially equivalent to the one found in [Sp1].
In order to state this main result we need the notion of combinatorial manifolds as follows:

Definition 1.1. *A simplicial complex M is called a* combinatorial manifold of dimension n *if the dimension of any simplex of M is at most n and if the link $lk(\Delta^k, M) := lk(\Delta^k)$ of any k-simplex Δ^k of M, $0 \leqslant k \leqslant n$, is a triangulated $(n-k-1)$-sphere.*
A diagonal is an edge consisting of vertices of M that itself does not belong to M.

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Two combinatorial manifolds (or more generally two simplicial complexes) M_1 and M_2 are said to be combinatorially equivalent, in this case we write $M_1 \sim M_2$, if there is a bijective mapping ϕ between the faces of M_1 and M_2 that is inclusion-preserving, i. e. such that $F_1 \subset F_2$ if and only if $\phi(F_1) \subset \phi(F_2)$.
The underlying complex of a combinatorial manifold (or a simplicial complex) M, written $|M|$, is the union of all simplices of M.

In the sequel the vertices of all examples will be denoted by $\langle i \rangle$ with $0 \leqslant i \leqslant 9$, $\langle a \rangle$ for the 11th and $\langle b \rangle$ for the 12th vertex. If the permutation $\zeta := (0, 1, \dots, 9, a, b)$ is an automorphism of the triangulation we regard, we talk about ζ as a cyclic symmetry generating the cyclic group C_{12}. Using the usual notation $f_i(M)$ (or simply f_i if there is no danger of confusion) for the number of i-faces of a combinatorial manifold M, $\chi(M)$ for its Euler-characteristic and the above definition we are now able to state our

Main Theorem 1.2. *There exist exactly three combinatorially distinct types of combinatorial 4-manifolds M_i, $1 \leqslant i \leqslant 3$, with ζ as an automorphism such that the 1-simplex $\langle 06 \rangle$ is a diagonal of M_i and such that $f_4(M_i) = 72 = 6 f_0(M_i) = 18\chi(M_i)$. For all the M_i it holds $|M_i| \simeq_{PL} S^2 \times S^2$, i. e. the M_i are PL-homeomorphic to $S^2 \times S^2$, and their automorphism groups $\mathrm{Aut}(M_i)$ are as follows:*

$$\mathrm{Aut}(M_i) \;\cong\; \begin{cases} C_{12} \rtimes C_2 & \textit{for } \; i = 1 \\ A_5 \rtimes C_4 & \textit{for } \; i = 2 \\ C_{12} \rtimes (C_2 \times C_2) & \textit{for } \; i = 3 \end{cases}$$

where $\rtimes$ denotes the semidirect product and C_k the cyclic group of order k.

Before proving this result in Section 2, we will now describe several interesting properties that are common to all M_i:
The f-vector of M_i is uniquely determined by the Dehn-Sommerville equations [Kü2] for triangulated 4-manifolds

$$2f_1 - 3f_2 + 4f_3 - 5f_4 = 0 = 2f_3 - 5f_4$$

to be

$$(f_0, f_1, f_2, f_3, f_4) = (12, 60, 160, 180, 72).$$

The cyclic symmetry immediately implies that the automorphism group of M_i is transitive on its vertices. This proves that the vertex links of all vertices are combinatorially equivalent.

The cyclic symmetry also yields in connection with the condition that $\langle 06 \rangle$ is a diagonal of M_i, that M_i has at least 6 pairwise disjoint diagonals. Hence M_i can be embedded in the 6-dimensional cross-polytope C_6^*. As $f_i(M_i) = f_i(C_6^*) = 2^{i+1}\binom{6}{i}$ for $i = 0, 1, 2$ any such embedding is 2-Hamiltonian, i. e. contains the 2-skeleton of C_6^*, and therefore M_i is simply-connected [Kü3, 3.8]. From these properties we

conclude that all M_i satisfy the assumptions of the following Lower Bound Theorem. Moreover they are examples for the case of equality in the inequality, hence they prove that the inequality is sharp:

Lower Bound Theorem 1.3 ([Sp2, 1.2])**.** *Assume M in $\mathbb{E}^d$ is a combinatorial $2k$-manifold, whose convex hull $\mathcal{H}(M) = P$ is – up to affine transformations – a centrally-symmetric simplicial d-polytope $P \subset \mathbb{E}^d$. Let M contain the k-skeleton $Sk_k(P)$ of P, that is the set of all faces of P of dimension at most k, and let M be a subcomplex of the boundary complex $\mathcal{C}(\partial P)$ of P. Then the following statements hold:*

(i) $(-1)^k \binom{2k+1}{k+1}(\chi(M) - 2) \geqslant 4^{k+1}\binom{\frac{1}{2}(d-1)}{k+1}$.

(ii) *For $d > 2k+1$ equality in (1) is attained if and only if P is affinely equivalent to the d-dimensional cross-polytope C_d^*.*

The M_i also satisfy equality in the next Upper Bound Theorem. A required fixed-point free involution is clearly given by ζ^6.

Upper Bound Theorem 1.4 ([Sp1, 4.1])**.** *Let M be a combinatorial 4-manifold with n vertices and with a fixed-point involution σ acting on the triangulation. Then $n = 2m$ is even and the inequality*

$$10(\chi(M) - 2) \leqslant \frac{4}{3}(m-1)(m-3)(m-5) = 4^3 \binom{\frac{1}{2}(m-1)}{3}.$$

holds with equality if and only if M can be embedded in C_m^* such that this embedding contains the 2-skeleton of C_m^*.

For the proof of the Main Theorem we are going to describe explicitly in the next section the triangulations M_i. From the lists given there it becomes obvious that they are all centrally-symmetric. In Section 3 we state further properties of the M_i. Finally in Section 4 we conclude with another interesting 12-vertex triangulation found while proving our result and we make some remarks concerned with 11-vertex triangulations of $S^2 \times S^2$.

2. Proof of the Main Theorem

`SUNI`, a computer program written by the first author and described in detail in [L1], is used for determining all possible candidates for 4-manifolds that satisfy several conditions. For instance all candidates found by `SUNI` are cyclic and satisfy the Dehn-Sommerville equations. Additionally a combinatorial test checks if the Euler-charateristic of all edge-links equals 2. In the case of our additional assumptions – in particular $\langle 06 \rangle$ has to be a diagonal – the program delivers three possible candidates:

(1)	1 1 1 1 8	2 2 3 2 3	1 1 2 1 7	1 4 3 2 2	1 1 3 5 2	1 3 1 3 4
(2)	1 1 1 1 8	2 2 3 2 3	1 1 2 1 7	1 4 3 2 2	1 1 3 5 2	1 3 1 4 3
(3)	1 1 1 1 8	2 2 3 2 3	1 1 2 5 3	1 1 3 5 2	1 1 3 4 3	1 2 1 4 4

The notation $(y_1, \dots, y_d)$ (or shorter $\boxed{y_1 \dots y_d}$) denotes a difference d-cycle, that generates a C_n-orbit of d-simplices as follows:

$$\left\{ \langle x\; x+y_1\; \dots\; x+\sum_{i=1}^{d-1} y_i \rangle \;\middle|\; x \in C_n \right\}.$$

(Note that $y_d := n - \sum_{i=1}^{d-1} y_i$.) In the case of Theorem 1.2 we have $d = 5$ and $n = f_0 = 12$. Let us denote candidate (i) by M_i and for the sake of brevity let us omit the symbols "$\langle$" and "$\rangle$" in the list of its simplices, then the complete list of 4-simplices of, for example, M_1 is given by

01234	02479	01245	0158a	0125a	01458
12345	1358a	12356	1269b	1236b	12569
23456	2469b	23467	237a0	23470	2367a
34567	357a0	34578	348b1	34581	3478b
45678	468b1	45689	45902	45692	45890
56789	57902	5679a	56a13	567a3	569a1
6789a	68a13	678ab	67b24	678b4	67ab2
789ab	79b24	789b0	78035	78905	78b03
89ab0	8a035	89a01	89146	89a16	89014
9ab01	9b146	9ab12	9a257	9ab27	9a125
ab012	a0257	ab023	ab368	ab038	ab236
b0123	b1368	b0134	b0479	b0149	b0347

The simplices of the other candidates can be calculated analogously from the difference-5-cycles. Obviously the M_i contain all edges except the diagonals $\langle j(j+6)\rangle$, where j as well as $j+6$ are regarded $\mod 12$. This is consistent with the theoretical value $f_1(M_i) = 60$.
All candidates are centrally-symmetric as $\langle a_1, \dots, a_j\rangle \in M_i$ implies $\langle b_1, \dots, b_j\rangle \in M_i$ where $b_l = a_l + 6 \mod 12$, $1 \leqslant l \leqslant j$.

It turns out that M_2 is combinatorially equivalent to the example M found in [Sp1, Theorem 3.1]. The images of the vertices of M (in the notation used in [Sp1]) of a required bijection $\phi : M \mapsto M_2$ are as follows:

$$\begin{array}{cccccc} \phi(0)=5, & \phi(1)=2, & \phi(2)=0, & \phi(3)=7, & \phi(4)=3, & \phi(5)=4, \\ \phi(\bar{0})=b, & \phi(\bar{1})=8, & \phi(\bar{2})=6, & \phi(\bar{3})=1, & \phi(\bar{4})=9, & \phi(\bar{5})=a. \end{array}$$

Unfortunately, the group $A_5 \times C_2$ discussed in [Sp1] is not the full automorphism group of M but only a subgroup of index 2 of it. The group given here is its full automorphism group.

For proving the Main Theorem it suffices to show the following Lemmata. The first two lemmata establish the classification and are therefore the most important part of the proof of 1.2. In the third lemma we calculate the automorphism groups.

Lemma 2.1. *$M_1 \not\simeq M_2 \not\simeq M_3 \not\simeq M_1$, i. e. there are three mutually combinatorial inequivalent triangulations.*

Lemma 2.2. *M_i, $1 \leqslant i \leqslant 3$, are combinatorial 4-manifolds with $|M_i| \simeq_{PL} S^2 \times S^2$.*

Lemma 2.3. $$\mathrm{Aut}(M_i) \cong \begin{cases} C_{12} \rtimes C_2 & \textit{for } i = 1 \\ A_5 \rtimes C_4 & \textit{for } i = 2 \\ C_{12} \rtimes (C_2 \times C_2) & \textit{for } i = 3 \end{cases}.$$

Proof of Lemma 2.1. The links of all edges of M_2 are subdivided octahedra as sketched in figure 1. For the other M_i this is not the case. Therefore $M_2 \not\sim M_v$ for $v \in \{1, 3\}$.

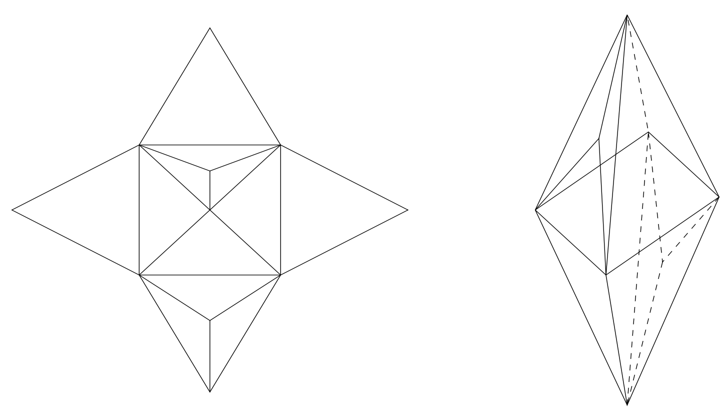

Figure 1. On the left-hand side the link of an edge of M_2, M_4 respectively is sketched and on the right-hand side its 3-dimensional realisation.

In the link $lk(k, M_i)$ of each edge k of M_i every vertex is adjacent to j other vertices where $3 \leqslant j \leqslant 6$. The number of vertices with j adjacent vertices in $lk(k, M_i)$ is denoted by $val(j)$ and

$$v(lk(k, M_i)) := (val(3), val(4), val(5), val(6))$$

is called the valence-vector of $lk(k, M_i)$. As the links of all vertices are combinatorially equivalent, it can only hold $M_{i_1} \sim M_{i_2}$, if for all $1 \leqslant j_1 \leqslant 5$ there exists a j_2 with $1 \leqslant j_1 \leqslant 5$ such that

$$v(\langle 0j_1 \rangle, M_{i_1}) = v(\langle 0j_2 \rangle, M_{i_2}).$$

But we get

$$v(\langle 03 \rangle, M_3) = (0, 4, 4, 0) \neq v(\langle 0j \rangle, M_1) = \begin{cases} (1, 3, 3, 1) & \text{for } j = 1 \\ (2, 0, 6, 0) & \text{for } j = 2 \\ (2, 2, 2, 2) & \text{for } j = 3 \\ (1, 4, 1, 2) & \text{for } j = 4 \\ (1, 3, 3, 1) & \text{for } j = 5 \end{cases}$$

Hence $M_1 \not\sim M_3$. $\square$

Proof of Lemma 2.2. As we have shown the statement for M_2 already in [Sp1] (up to combinatorial equivalence, cf. page 4) and as the proofs for M_1 and M_3 are similar, but not completely analogous to the one given in [Sp1] we will present here a proof only for M_3.

As $\mathrm{Aut}(M_3)$ operates transitively on the vertices if suffices to show that the link $lk(0)$ of vertex 0 is a triangulated S^3. To do this we have to establish:

(i) The link of each vertex of $lk(0)$, i. e. the link of each edge containing vertex $\langle 0\rangle$, is a triangulated 2-sphere. Because of automorphism ζ we only have to check this for the vertices $\langle i\rangle$ with $1 \leqslant i \leqslant 5$. Here we can sketch the triangulations in the same way as shown in Figure 1 and we get 2-spheres with the valence vector we have calculated in the preceding lemma.

(ii) $lk(0)$ is a triangulated 3-sphere. To show this we split up the vertex set of $lk(0)$ into two disjoint subset and regard their span:

$$A := \mathit{span}(1,2,3,4,7) \text{ and } B := \mathit{span}(5,8,9,a,b).$$

A and B each consist of two tetrahedra and are therefore collapsible. Hence A and B are 3-balls (cf. [R-S]). Cluing these balls together proves that the underlying complex of $lk(0)$ is homeomorphic to S^3.

It remains to determine the topological type of $|M_3|$. Let

$$\alpha_1 := \mathit{span}(0,2,4,6,8,a) \quad\text{and}\quad \alpha_2 := \mathit{span}(0,1,3,5).$$

Then $\alpha_1 \sim \partial C_3^*$ and $\alpha_2 \sim \Delta^3$. It turns out that α_1 together with α_2 generate the second homology group $H_2(M_3,\mathbb{Z})$ and that the intersection form of M_3 equals the intersection form of $S^2 \times S^2$, namely

$$\begin{pmatrix} \alpha_1\cdot\alpha_1 & \alpha_1\cdot\alpha_2 \\ \alpha_2\cdot\alpha_1 & \alpha_2\cdot\alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This is an even form and by Freedman's Theorem [Fr] there exists exactly one simply-connected, closed, topological 4-manifold representing that form. Hence $|M_3|$ must be homeomorphic to $S^2 \times S^2$.
(To get for instance the intersection number $\alpha_1 \cdot \alpha_2$ regard

$$N_1 := \mathit{span}(0,1,3,5) \text{ and } N_2 := \mathit{span}(0,2,4,6,7,8,9,a,b).$$

As

$$lk(0) \setminus \mathrm{span}(0,1,3,5) = lk(0) \setminus (N_1 \cap lk(0)) \searrow N_2 \cap lk(0)$$

(where "$X \searrow Y$" means "X collapses onto Y" [R-S]), the complement of $N_1 \cap lk(0)$ is homotopically equivalent to $N_2 \cap lk(0)$. Because

$$|lk(0) \setminus \mathrm{span}(0,1,3,5)| = \langle 13\rangle \cup \langle 35\rangle \cup \langle 51\rangle$$

and

$$|N_2 \cap lk(0)| \searrow \langle 24\rangle \cup \langle 48\rangle \cup \langle 8a\rangle \cup \langle a2\rangle$$

we get two disjoint cycles that must be unknotted and linked in $lk(0)$. Consequently $\alpha_1 \cdot \alpha_2 = \pm 1$ and with an appropriate orientation we get the intersection number stated above.)
To prove that $|M_3|$ is even PL-homeomorphic to $S^2 \times S^2$ we can mimic the prove given in [Sp1] that essentially uses that there is only one 3-ball bundle over S^2 with

the intersection form of $S^2 \times S^2$. This proof does not use Freedman's Theorem but we still need to calculate the intersection form. $\square$

Proof of Lemma 2.3. We use a simple GAP-program[1] written by the second author. This program explicitly calculates all bijections between the vertex-links of two candidates. The order of $\mathrm{Aut}(lk(0, M_i))$ is 2 for $i = 1$ and 4 for $i = 3$. The automorphisms of these cases are

$$id,\ \alpha = (1,5)(2,a)(4,8)(7,b) \quad \text{for} \quad i = 1$$

and

$$id,\ \beta_1 = (1,b)(2,a)(3,9)(4,8)(5,7),\ \beta_2 = (1,7)(3,9)(5,b),$$

$$\beta_3 = (1,5)(2,a)(4,8)(7,b) \quad \text{for} \quad i = 3.$$

For $i = 2$ we get 20 automorphisms. In all cases it is easy to show

$$\mathrm{Aut}(lk(0, M_i)) = \begin{cases} C_2 & \text{for } i = 1 \\ C_5 \rtimes C_4 & \text{for } i = 2 \\ C_2 \times C_2 = A^{(2,2)} & \text{for } i = 3 \end{cases},$$

where $A^{(2,2)}$ denotes the Kleinian group.
(Note that $\mathrm{Aut}(lk(0, M_2))$ is transitive on the vertices of $lk(0, M_2)$ because

$$(2,a,b,9,7)(1,8,4,5,3) \in \mathrm{Aut}(lk(0, M_2)) \ni (1,a,8,9)(2,3,7,4)(5,b).$$

By the cyclic symmetry $\mathrm{Aut}(M_2)$ acts therefore transitively on the edges of M_2. This is not the case for $i = 1$ and $i = 3$.)
The transitivity of $\mathrm{Aut}(M_i)$ on the vertices leads to

$$|\mathrm{Aut}(M_i)| \leqslant f_0(M_i)|\mathrm{Aut}(lk(0, M_i))| = \begin{cases} 24 & \text{for } i = 1 \\ 240 & \text{for } i = 2 \\ 48 & \text{für } i = 3 \end{cases}. \tag{1}$$

Denote by $A \cdot B$ the complex product of two groups A and B.
As (i) $|\langle\zeta\rangle \cdot \langle\alpha\rangle| \cong |C_{12} \cdot C_2| = 24$, (ii) $\alpha \in \mathrm{Aut}(M_1)$ and (iii) $C_{12} \triangleleft C_{12} \cdot C_2$, inequality (1) implies that $C_{12} \cdot C_2 \cong C_{12} \rtimes C_2$ is the full automorphism group of M_1.

Analogously it can be shown $\mathrm{Aut}(M_3) \cong C_{12} \cdot \langle \beta_1, \beta_2, \beta_3 \rangle$. $\mathrm{Aut}(lk(0, M_3))$ is not normal in $\mathrm{Aut}(M_3)$ as

$$\zeta \cdot \langle \mathrm{Aut}(lk(0, M_3)) \rangle \neq \langle \mathrm{Aut}(lk(0, M_3)) \rangle \cdot \zeta,$$

but

$$\beta_i \langle\zeta\rangle = \langle\zeta\rangle \beta_i \quad \text{for} \quad 1 \leqslant i \leqslant 3$$

and therefore $\langle\zeta\rangle \triangleleft \mathrm{Aut}(M_3)$. Hence $\mathrm{Aut}(M_3) \cong C_{12} \rtimes A^{(2,2)}$.

It remains to discuss the automorphism group of M_2. $A_5 \times C_2$ is a subgroup of $\mathrm{Aut}(M_2)$ as already seen in [Sp1]. This subgroup is generated by the permutations

$$(0,7,3,4,2)(1,9,a,8,6), \quad (0,6)(1,7)(2,8)(3,9)(4,a)(5,b)$$

[1] More information on the freeware-algebra-system GAP can be found under the URL `http://www-gap.dcs.st-and.ac.uk/~gap/`. For information on the GAP-program used here, send an E-Mail to the second author.

$$\text{and} \qquad (0,6)(1,9)(2,5)(3,7)(4,a)(8,b).$$

(Here we had to rename the vertices according to the bijection ϕ stated above.) Additionally ζ is a (cyclic) automorphism operating on M_2. According to GAP the order of the group generated by these four permutations is 240 and therefore by (1) it must be the full automorphism group. A closer examination shows $\mathrm{Aut}(M_2) \cong A_5 \rtimes C_4$. For the center of this group we get $Z(\mathrm{Aut}(M_2)) \cong C_2$ and its system of normal subgroups can be seen in the next figure.

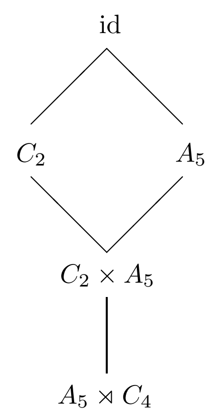

Figure 2. Normal subgroups of $\mathrm{Aut}(M_2)$.

Furthermore it holds $\mathrm{Aut}(M_2)/Z(\mathrm{Aut}(M_2)) \cong S_5$, but S_5 is not a subgroup of $\mathrm{Aut}(M_2)$. □

Note that Lemma 2.3 immediately yields $M_1 \not\sim M_2 \not\sim M_3 \not\sim M_1$.

Remark 2.4. *The program* SUNI *has found the following three other candidates:*

(4)	1 1 1 1 8	2 2 3 2 3	1 2 2 3 4	1 1 7 1 2	1 1 2 5 3	1 3 1 3 4
(5)	1 1 1 1 8	2 2 3 2 3	1 2 2 3 4	1 1 7 1 2	1 1 2 5 3	1 3 1 4 3
(6)	1 1 2 1 7	1 4 3 2 2	1 2 2 3 4	1 1 7 1 2	1 1 2 5 3	1 1 3 5 2

We have $M_1 \sim M_5$ and $M_2 \sim M_4$ as can be seen immediately be applying the multiplier $-1 \equiv 11 \mod 12$ on M_1, M_2 respectively. (For the notion of multipliers see Section 3.*)*
This can also be proved by the GAP*-program mentioned in the proof of* 2.2*. Any of the calculated bijections between the vertex links of two examples can be extended to give a bijection between the examples themselves.*

Example M_6 is not a manifold as we prove now:

As $\langle 048 \rangle \notin M_6$, M_6 is not 2-Hamiltonian in C_6^* and therefore not a triangulated manifold. (An upgrade of the program SUNI *meanwhile also calculates the f-vector of all candidates. The output of this extension does not contain M_6 anymore.)*

Alternatively, one can also examine the link $lk(\langle 04\rangle)$ *of edge* $\langle 04\rangle$*. It consists of the following* 2*-simplices:*

$$\begin{array}{cccccc} \langle 125\rangle, & \langle 13b\rangle, & \langle 79b\rangle, & \langle 259\rangle, & \langle 27b\rangle, & \langle 579\rangle, \\ \langle 135\rangle, & \langle 23b\rangle, & \langle 129\rangle, & \langle 357\rangle, & \langle 19b\rangle, & \langle 237\rangle. \end{array}$$

Therefore $f_0(lk(\langle 04\rangle)) = 7$*,* $f_1(lk(\langle 04\rangle)) = 18$*,* $f_2(lk(\langle 04\rangle)) = 12$ *and* $\chi(lk(\langle 04\rangle)) = 1$*. Hence* M_6 *is not even an Eulerian* 4*-manifold, i. e.* $\chi(lk(\Delta^k)) = 1-(-1)^{4-k} = 1-(-1)^k$ *is not satisfied for all* k*-simplices of* M_6*.*
But $lk(\langle 04\rangle)$ *is a pinched* 2*-sphere and therefore a* 2*-manifold with one singularity as can be seen from Figure* 3*.*

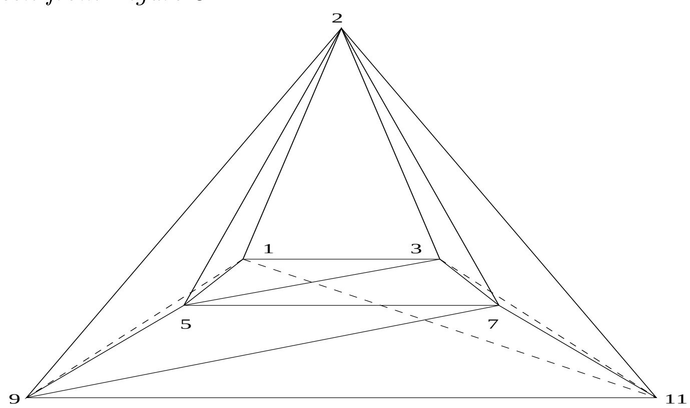

Figure 3. $lk(\langle 04\rangle)$ *of example* M_6

3. Properties of the triangulations of $S^2 \times S^2$

In this section we describe further properties of the examples M_i. We begin with a simple observation.

The span of the "even" vertices as well as the span of the "odd" vertices of all M_i is a subcomplex of the octahedron, i. e. the 3-cross-polytope. This holds because e. g. $span(0,2,4,6,8,a)$ contains no k-simplex with $k \geqslant 3$. (Otherwise there would be a 3-simplex in $span(0,2,4,6,8,a)$ containing a diagonal as an edge.) The f-vector of the M_i implies, that in $span(0,2,4,6,8,a)$ all 2-simplices without one of the diagonals $\langle 06\rangle$, $\langle 28\rangle$ and $\langle 4a\rangle$ are contained. Therefore we get that $span(0,2,4,6,8,a)$ equals the boundary complex of C_3^*. Analogously we get the same result for the odd vertices.
This not only holds for the even and odd vertices, but for all subsets S of the vertex set with $|S| = 6$, such that 3 diagonals are contained in the regarded subset.

Another interesting subset is the cylinder C consisting of the triangles

$$\langle 024\rangle,\ \langle 246\rangle,\ \langle 468\rangle,\ \langle 68a\rangle,\ \langle 8a0\rangle \text{ and } \langle 0a2\rangle.$$

This cylinder is invariant under automorphism ζ^2. Together with the triangles $\langle 048\rangle$ and $\langle 26a\rangle$, whose boundaries form the boundary of C, we get $span(0,2,4,6,8,a)$. Likewise there is a cylinder in $span(1,3,5,7,9,b)$.

Now let us regard $lk(0)$ of example M_2. The vertices of this link can be subdivided into two subsets S_{2_1} and S_{2_2} with each 5 elements, such that the span of both subsets form Möbius strips in $lk(0)$ (cf. [Sp1]). As all vertex links are combinatorally equivalent, the same holds for any vertex link.
It is easy to show that the required subdivision is unique: Because any 5-vertex Möbius strip must be neighborly, no two elements of S_{2_1} may form a diagonal of M_2. Suppose without loss of generality $1 \in S_{2_1}$. Then $7 \in S_{2_2}$ and there remain $8 \cdot 6 \cdot 4 \cdot 2 = 384$ possibilities to check.
In [Sp1] we used these Möbius strips to compute the intersection numbers. In this case however it was not necessary that we got Möbius strips, but that the Möbius strips were linked and unknotted. So although there is no subdivision of any vertex link of either M_1 or M_3, that also produces disjoint Möbius strips, the proof still carries over to the cases M_1 and M_3. Nonetheless the existence of disjoint Möbius strips stills "shows" that example M_2 is the most symmetric one.

Inequality (1) gives an upper bound on $|\mathrm{Aut}(M_i)|$. We will derive now a lower bound. For this we use the notion of multipliers.

Definition 3.1. *Let the vertices of a simplical complex K be numbered by* $0, \dots,$ $k-1$. *$a \in C_k$ is called a multiplier of K if*

$$\mu_k : C_k \to C_k, \quad i \mapsto i \cdot a \mod k$$

is an automorphism of K.

Recall the following simple facts:

(i) If a is a multiplier of K, then $gcd(a,k) = 1$.
(ii) If $(0, \dots, k-1)$ is an automorphism of K and if K has l different multipliers, then $|\mathrm{Aut}(K)| \geqslant l \cdot k$.

From (i) we know that the only possible multipliers of our triangulations are $1, 5, 7$ and 11. M_1 and M_2 have multipliers 1 and 5, whereas under the maps μ_7 and μ_{11} the image of M_1 is M_2 and vice versa. M_3 has all multipliers. Hence by (ii)

$$|\mathrm{Aut}(M_i)| \geqslant \begin{cases} 24 & \text{for } i = 1,\ 2 \\ 48 & \text{for } i = 3 \end{cases} .$$

Together with (1) we conclude

$$|\mathrm{Aut}(M_i)| = \begin{cases} 24 & \text{for } i = 1 \\ 48 & \text{for } i = 3 \end{cases} .$$

4. Further 12- and 11-vertex triangulations

In the next remark we describe another interesting simplical complex found by `SUNI`.

Remark 4.1. *The program* `SUNI` *has delivered another candidate M for being a 4-manifold with* 12 *vertices and with cyclic symmetry. In this example we dropped*

the additional assumption that $\langle 06 \rangle$ has to be a diagonal. This example is rather interesting as it is highly-symmetric, as one can see from its invariance under all *multipliers. It is given by the difference-5-cycles*

$$1\,1\,1\,3\,6 \quad 1\,4\,2\,3\,2 \quad 1\,2\,3\,2\,4 \quad 1\,1\,1\,6\,3 \quad 1\,1\,3\,3\,4 \quad 1\,3\,1\,5\,2 \quad 1\,2\,5\,1\,3 \quad 1\,1\,4\,3\,3$$

and we get for the f-vector $(f_1, \ldots, f_4) = (12, 66, 204, 240, 96)$ and consequently $\chi(M) = 6$. (Missing triangles are for instance $\langle 024 \rangle$ and $\langle 048 \rangle$; cp. the properties of M_1 - M_3 as described on the previous page.) However this candidate is not a 4-manifold, as the link of the edge $\langle 01 \rangle$ consists of three 2-spheres that are only edge- but not vertex-disjoint. These spheres are clued together at their poles ("Banana-surface") as shown in the next figure.

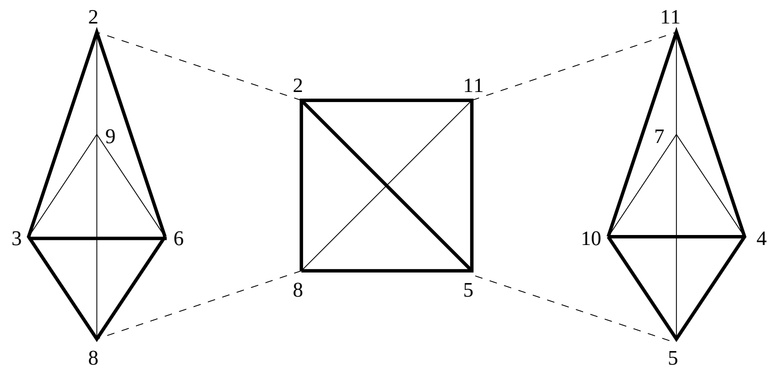

Figure 3. *The link of edge $\langle 01 \rangle$.*

But M is at least an Eulerian 4-manifold [Kü2] *as follows from calculating the Euler-characteristic of the links of all simplices Δ^k of the triangulation. Because of the cyclic symmetry it suffices to show $\chi(lk(\langle i \rangle)) = 0$ for one vertex.* `SUNI` *itself confirms $\chi(lk(\Delta^1)) = 2$ for all 1-simplices of the triangulation and finally $\chi(lk(\Delta^2)) = 0 = 2 - \chi(lk(\Delta^3))$ can easily be checked.*
The singular locus of this example is the union of two tori. One of these is given by the difference-3-cycles 012 and $05a \mod 12$, the other by 015 and 045. Each of these tori is invariant under the group of order 48 generated by the cyclic automorphism ζ and the four multipliers μ_k.

Remark 4.2. *Homology investigations of Frank Lutz (oral communication) imply that their are no other 12-vertex triangulations of $S^2 \times S^2$ with an automorphism group that is transitive on the vertices than the examples found in our Main Theorem. As these are all rather symmetric (especially example M_2), we felt that it will be possible to "break" a bit of the symmetry of at least one of our examples to construct a triangulation of $S^2 \times S^2$ with just 11 vertices.*
That the theoretical lower bound on the number of vertices according to [Kü3] *is 10 and that this value was ruled out only by an extensive computer search made this assumption even more plausible.*
While preparing this paper an unpublished `GAP`*-program written by Frank Lutz shows that there really is such an example:*

There does exist a triangulation of $S^2 \times S^2$ with 11 vertices – the minimum number of vertices.

It remains open if this 11*-vertex-triangulation is unique. We conjecture that this is not the case. Using bistellar operations one can possibly construct* $S^2 \times S^2$ *with* 11 *vertices not combinatorially equivalent to the given triangulation.*

Apart from a computer search as applied by Frank Lutz, a more systematic way of constructing such a triangulation may be to start with one of our examples M_i *and cut off the open star of one vertex. The resulting complex* $\widetilde{M}_i$ *has* 11 *vertices and is homeomorphic to* $S^2 \times S^2$ *with an open ball removed. Now try to close* $\widetilde{M}_i$ *by inserting new simplices, but no new vertices. Although this procedure may not always be possible, it can often be carried out* [ABS]*. Here we may use simplices that contain edges that are diagonals in* M_i*; this is the process we refered to as "breaking of symmetry". Finding such simplices naturally leads a system of linear equations.*

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